

GENERALIZED POSITIVE LINEAR OPERATORS BASED ON PED AND IPED

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ABSTRACT. The paper deals with generalized positive linear operators based on Pólya-Eggenberger distribution(PED) as well as inverse Pólya-Eggenberger distribution(IPED). Initially, we give the moments by using Stirling numbers of second kind and then establish direct results, convergence with the help of local and weighted approximation of proposed operators.

1. INTRODUCTION

In the year 1968, D. D. Stancu [23] introduced a new class of positive linear operators based on Pólya-Eggenberger distribution(PED) and associated to a real-valued function on $[0, 1]$ as:

$$B_n^{(\alpha)}(f; x) = \sum_{k=0}^n \binom{n}{k} \frac{x^{[k, -\alpha]}(1-x)^{[n-k, -\alpha]}}{1^{[n, -\alpha]}} f\left(\frac{k}{n}\right), \quad (1.1)$$

where α is a non-negative parameter which may depend only on the natural number n and $t^{[n, h]} = t(t-h)(t-2h) \cdots (t-\overline{n-1}h)$, $t^{[0, h]} = 1$ represents the factorial power of t with increment h .

Later on Stancu [24] introduced a generalized form of the Baskakov operators based on inverse Pólya-Eggenberger distribution(IPED) for a real-valued function bounded on $[0, \infty)$, given by

$$V_n^{(\alpha)}(f; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1^{[n, -\alpha]}x^{[k, -\alpha]}}{(1+x)^{[n+k, -\alpha]}} f\left(\frac{k}{n}\right). \quad (1.2)$$

Now we consider new positive linear operators $L_n^{(\alpha)}$, for each f , real valued function bounded on interval I , as:

$$L_n^{(\alpha)}(f; x) = \sum_k w_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \quad x \in I, \quad n = 1, 2, \dots, \quad (1.3)$$

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where $\alpha = \alpha(n) \rightarrow 0$ when $n \rightarrow \infty$, p & k are nonnegative integers and for $\lambda = -1, 0$, we have

$$\begin{aligned}\omega_{n,k}^{(\alpha)}(x) &= \frac{n+p}{n+p+\overline{\lambda+1}k} \binom{n+p+\overline{\lambda+1}k}{k} \frac{\prod_{i=0}^{k-1} (x+i\alpha) \prod_{i=0}^{n+p+\lambda k-1} (1+\lambda x+i\alpha)}{\prod_{i=0}^{n+p+\overline{\lambda+1}k-1} (1+\overline{\lambda+1}x+i\alpha)} \\ &= \frac{n+p}{n+p+\overline{\lambda+1}k} \binom{n+p+\overline{\lambda+1}k}{k} \frac{x^{[k,-\alpha]}(1+\lambda x)^{[n+p+\lambda k,-\alpha]}}{(1+\overline{\lambda+1}x)^{[n+p+\overline{\lambda+1}k,-\alpha]}},\end{aligned}$$

by using the notation $\overline{m-r}\alpha = (m-r)\alpha$. Operators (1.3) is generalized form of above two operators (1.1) and (1.2) and associated with PED and IPED [7].

Kantorovich form of Stancu operators (1.1) had been given by Razi [21] and he studied its convergence properties and degree of approximation. Ispir et al. [13] also discussed the Kantorovich form of operators (1.1) and they estimated the rate of convergence for absolutely continuous functions having a derivative coinciding a.e. with a function of bounded variation. Recently, Miclaus [16] established some approximation results for the Stancu operators (1.1) and its Durrmeyer type integral modification was studied by Gupta and Rassias [12] and obtained some direct results which include an asymptotic formula, local and global approximation results for these operators in terms of modulus of continuity.

Very recently Durrmeyer type modification of generalized Baskakov operators (1.2) associated with IPED were introduced by Dhamija and Deo [6] and studied the moments with the help of Vandermonde convolution formula and then gave approximation properties of these operators which include uniform convergence and degree of approximation. Deo et al. [4] also investigated approximation properties of Kantorovich variant of operators (1.2) and they established uniform convergence, asymptotic formula and degree of approximation. Various Durrmeyer type modifications and then their local as well as weighted approximation along with some other approximation behaviour have been discussed by many authors e.g., ([3], [14], [11]).

The main object of this paper is to find moments with the help of Stirling numbers of second kind and estimate the rate of convergence of operators (1.3) by studying local and weighted approximation theorems.

Throughout this paper we consider interval $I = [0, \infty)$ for $\lambda = 0$ and $I = [0, 1]$ for $\lambda = -1$.

2. SPECIAL CASES

It is easy to understand that the special cases of operators (1.3) are as follows:

- (1) For $\lambda = -1$; we have
 - (i) When $\alpha \neq 0 \neq p$

$$L_n^{(\alpha)}(f; x) = \sum_{k=0}^{n+p} \binom{n+p}{k} \frac{\prod_{i=0}^{k-1} (x+i\alpha) \prod_{i=0}^{n+p-k-1} (1-x+i\alpha)}{\prod_{i=0}^{n+p-1} (1+i\alpha)} f\left(\frac{k}{n}\right).$$

This leads to Schurer type Stancu operators.

(ii) When $\alpha \neq 0, p = 0$ we get alternate form of operators (1.1) as:

$$L_n^{(\alpha)}(f; x) = \sum_{k=0}^n \binom{n}{k} \frac{\prod_{i=0}^{k-1} (x + i\alpha) \prod_{i=0}^{n-k-1} (1 - x + i\alpha)}{\prod_{i=0}^{n-1} (1 + i\alpha)} f\left(\frac{k}{n}\right).$$

Particular case: when $\alpha = 1/n, p = 0$ we obtain Lupaş and Lupaş operators [15] as:

$$L_n^{(\alpha)}(f; x) = \frac{2(n!)}{(2n)!} \sum_{k=0}^n \binom{n}{k} \prod_{i=0}^{k-1} (nx + i) \prod_{i=0}^{n-k-1} (\overline{1 - xn} + i) f\left(\frac{k}{n}\right).$$

(iii) When $\alpha = 0, p \neq 0$ we have Bernstein-Schurer operators [22] as:

$$L_n^{(\alpha)}(f; x) = \sum_{k=0}^{n+p} \binom{n+p}{k} x^k (1-x)^{n+p-k} f\left(\frac{k}{n}\right).$$

(iv) When $\alpha = 0, p = 0$ we obtain original Bernstein operators [2] as:

$$L_n^{(\alpha)}(f; x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

(2) For $\lambda = 0$; we obtain the following operators:

(i) When $\alpha \neq 0, p \neq 0$ we have Stancu-Schurer operators as:

$$L_n^{(\alpha)}(f; x) = \sum_{k=0}^{\infty} \binom{n+p+k-1}{k} \frac{\prod_{i=0}^{k-1} (x + i\alpha) \prod_{i=0}^{n+p-1} (1 + i\alpha)}{\prod_{i=0}^{n+p+k-1} (1 + x + i\alpha)} f\left(\frac{k}{n}\right),$$

(ii) When $\alpha \neq 0, p = 0$ we obtain alternate form of operators (1.2) as:

$$L_n^{(\alpha)}(f; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\prod_{i=0}^{k-1} (x + i\alpha) \prod_{i=0}^{n-1} (1 + i\alpha)}{\prod_{i=0}^{n+k-1} (1 + x + i\alpha)} f\left(\frac{k}{n}\right),$$

(iii) When $\alpha = 0, p \neq 0$ we get Baskakov-Schurer operators as:

$$L_n^{(\alpha)}(f; x) = \sum_{k=0}^{\infty} \binom{n+p+k-1}{k} \frac{x^k}{(1+x)^{n+p+k}} f\left(\frac{k}{n}\right),$$

(iv) When $\alpha = 0, p = 0$ we obtain classical Baskakov operators [1] as:

$$L_n^{(\alpha)}(f; x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} f\left(\frac{k}{n}\right).$$

3. PRELIMINARY RESULTS

In 1730, J. Stirling [25] introduced an important concept of numbers, useful in various branches of mathematics like number theory, calculus of Bernstein polynomials etc., known as Stirling numbers of first kind and afterwards Stirling numbers of second kind. Let \mathbb{R} be the set of real numbers and \mathbb{N} , a collection of natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $x \in \mathbb{R}$ and $i, j \in \mathbb{N}_0$, let $S(j, i)$ denote the Stirling numbers of second kind then

$$x^j = \sum_{i=0}^j S(j, i) (x)_i,$$

with alternate form

$$x^j = \sum_{i=0}^{j-1} S(j, j-i) (x)_{j-i}, \quad (3.1)$$

and has the following properties:

$$S(j, i) := \begin{cases} 1, & \text{if } j = i = 0; j = i \text{ or } j > 1, i = 1 \\ 0, & \text{if } j > 0, i = 0 \\ 0, & \text{if } j < i \\ i.S(j-1, i) + S(j-1, i-1), & \text{if } j, i > 1. \end{cases} \quad (3.2)$$

These Stirling numbers of second kind are very useful in calculating the moments of linear positive operators especially for higher order moments. In [17, 18], Dan obtained higher order moments for Bernstein type operators by using these numbers. In what follows, we shall find the moments of $L_n^{(\alpha)}$ given by (1.3) with the help of same. Let us recall the monomials $e_j(x) = x^j, j \in \mathbb{N}_0$ be the test functions.

Lemma 3.1. *For the monomial t^j , where $j \in \mathbb{N}$, and $t, x \in I$, we have*

$$L_n^{(\alpha)}(t^j; x) = \frac{1}{n^j} \sum_{i=0}^{j-1} S(j, j-i) \phi_{n+p}^{j-i} \frac{x^{[j-i, -\alpha]}}{1^{[j-i+(\lambda+1), -\alpha]}},$$

where

$$\phi_{n+p}^{j-i} = \begin{cases} (n+p)_{(j-i)}, & \lambda = -1 \\ (n+p)^{(j-i)}, & \lambda = 0 \end{cases},$$

$(y)_n := \prod_{i=0}^{n-1} (y-i), (y)_0 := 1$ and $(y)^n := \prod_{i=0}^{n-1} (y+i), (y)^{(0)} := 1$ are respectively the falling factorial and rising factorial with $y \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof. Using the relation (3.1), it is easy to derive above expression on the same lines as in [19] (Page 54, Theorem 2.1.94.). \square

Lemma 3.2. *For the Generalized positive linear operators (1.3) hold*

$$\begin{aligned} L_n^{(\alpha)}(1; x) &= 1, \quad L_n^{(\alpha)}(t; x) = \left(\frac{n+p}{n} \right) \frac{x}{(1-\lambda+1\alpha)}, \\ L_n^{(\alpha)}(t^2; x) &= \left(\frac{n+p}{n^2} \right) \frac{1}{(1-\lambda\alpha)(1-\bar{\lambda}+1\alpha)} \\ &\quad \times \left[\frac{(n+p+\lambda+1)x(x+\alpha)}{1-2\bar{\lambda}+1\alpha} + x(1+\lambda x) \right], \end{aligned}$$

$$L_n^{(\alpha)}(t^3; x) = \frac{(n+p)x}{n^3(1-\overline{\lambda+1\alpha})} \left[\frac{(n+p+2\lambda+1)(n+p+2\overline{2\lambda+1})(x+\alpha)(x+2\alpha)}{(1-3\lambda+2\alpha)(1-5\lambda+3\alpha)} + \frac{3(n+p+2\lambda+1)(x+\alpha)}{(1-3\lambda+2\alpha)} + 1 \right],$$

and

$$L_n^{(\alpha)}(t^4; x) = \frac{(n+p)x}{n^4(1-\overline{\lambda+1\alpha})} \times \left[\frac{(n+p+2\lambda+1)(n+p+2\overline{2\lambda+1})(n+p+3\overline{2\lambda+1})(x+\alpha)(x+2\alpha)(x+3\alpha)}{(1-3\lambda+2\alpha)(1-5\lambda+3\alpha)(1-7\lambda+4\alpha)} + \frac{6(n+p+2\lambda+1)(n+p+2\overline{2\lambda+1})(x+\alpha)(x+2\alpha)}{(1-3\lambda+2\alpha)(1-5\lambda+3\alpha)} + \frac{7(n+p+2\lambda+1)(x+\alpha)}{(1-3\lambda+2\alpha)} + 1 \right].$$

Proof. From the definition of operators (1.3), we can obtain the moment for $j = 0$, i.e. $L_n^{(\alpha)}(1; x) = 1$

Also by the application of Lemma 3.1 for $j = 1, 2, 3, 4$ and taking into account the relation (3.2), we can follow the values of remaining moments. \square

Further, to obtain the central moments of generalized positive operators (1.3), we use the following result:

Lemma 3.3. [10] *Let V be any linear operators then*

$$V\left((t-x)^j; x\right) = V\left(t^j; x\right) - \sum_{i=0}^{j-1} \binom{j}{i} x^{j-i} V\left((t-x)^i; x\right),$$

and in the case when $V(t^j; x) = x^j$, for $j = 0, 1$, then we get

$$V\left((t-x)^3; x\right) = V\left(t^3; x\right) - x^3 - 3xV\left((t-x)^2; x\right),$$

and

$$V\left((t-x)^4; x\right) = V\left(t^4; x\right) - x^4 - 4xV\left((t-x)^3; x\right) + 6x^2V\left((t-x)^2; x\right).$$

Lemma 3.4. *The generalized linear positive operators (1.3) satisfy*

$$L_n^{(\alpha)}(t-x; x) = \frac{px + \overline{\lambda+1\alpha}}{n(1-\overline{\lambda+1\alpha})}, \quad (3.3)$$

$$L_n^{(\alpha)}((t-x)^2; x) = \frac{n+p}{n(1-\lambda\alpha)(1-\overline{\lambda+1\alpha})} \left[(1-\lambda\alpha)(1-\overline{\lambda+1\alpha}) \frac{nx^2}{n+p} + \frac{(n+p+\lambda+1)x(x+\alpha)}{n(1-2\overline{\lambda+1\alpha})} + \frac{x(1+\lambda x)}{n} - 2(1-\lambda\alpha)x^2 \right], \quad (3.4)$$

$$\begin{aligned}
L_n^{(\alpha)} \left((t-x)^3; x \right) &= \frac{(n+p)x(x+\alpha)}{n^2(1-\overline{\lambda+1}\alpha)} \left[\frac{(n+p+2\lambda+1)(n+p+2\overline{2\lambda+1})(x+2\alpha)}{n(1-\overline{3\lambda+2}\alpha)(1-\overline{5\lambda+3}\alpha)} \right. \\
&\quad \left. + \frac{3(n+p+2\lambda+1)}{n(1-\overline{3\lambda+2}\alpha)} - \frac{3(n+p+\lambda+1)x}{(1-\lambda\alpha)(1-2\overline{(\lambda+1)\alpha})} \right] \\
&\quad + \frac{(n+p)x}{n(1-\overline{\lambda+1}\alpha)} \left[\frac{1}{n^2} - 3x \left(\frac{px+\overline{\lambda+1}\alpha}{(n+p)} \right) - \frac{3}{(1-\lambda\alpha)} \frac{x(1+\lambda x)}{n} \right] \\
&\quad - 2x^3 \left[2 - \frac{3(n+p)}{n(1-\overline{\lambda+1}\alpha)} \right], \tag{3.5}
\end{aligned}$$

$$\begin{aligned}
L_n^{(\alpha)} \left((t-x)^4; x \right) &= \frac{(n+p)x(x+\alpha)}{n^2(1-\overline{\lambda+1}\alpha)} \\
&\quad \times \left[\frac{(n+p+2\lambda+1)(n+p+2\overline{2\lambda+1})(n+p+3\overline{2\lambda+1})(x+2\alpha)(x+3\alpha)}{n^2(1-\overline{3\lambda+2}\alpha)(1-\overline{5\lambda+3}\alpha)(1-\overline{7\lambda+4}\alpha)} \right. \\
&\quad + \frac{2(3-2nx)(n+p+2\lambda+1)(n+p+2\overline{2\lambda+1})(x+2\alpha)}{n^2(1-\overline{3\lambda+2}\alpha)(1-\overline{5\lambda+3}\alpha)} \\
&\quad \left. + \frac{(7-12nx)(n+p+2\lambda+1)}{n^2(1-\overline{3\lambda+2}\alpha)} + \frac{6(n+p+\lambda+1)x^2}{(1-\lambda\alpha)(1-2\overline{(\lambda+1)\alpha})} \right] \\
&\quad + \frac{(n+p)x}{n(1-\overline{\lambda+1}\alpha)} \left[\frac{1-4nx}{n^3} + 8x^2 \left(\frac{px+\overline{\lambda+1}\alpha}{(n+p)} \right) \frac{6x^2(1+\lambda x)}{n(1-\lambda\alpha)} \right] \\
&\quad + 3x^4 \left[3 - \frac{4(n+p)}{n(1-\overline{\lambda+1}\alpha)} \right]. \tag{3.6}
\end{aligned}$$

Proof. The combined use of Lemma 3.2 and 3.3 will follow the proof. \square

Remark 3.1. For sufficiently large n , Lemma 3.5 gives the following inequalities:

$$\begin{aligned}
\text{(i)} \quad L_n^{(\alpha)} \left((t-x)^2; x \right) &\leq A_1 \frac{(1+x^2)}{n}, \\
\text{(ii)} \quad L_n^{(\alpha)} \left((t-x)^4; x \right) &\leq A_2 \frac{(1+x^2)^2}{n},
\end{aligned}$$

where A_1 and A_2 are some positive constants.

Lemma 3.5. For positive linear operators (1.3), there holds,

$$|L_n^{(\alpha)}(f; x)| \leq \|f\|.$$

Proof. From operators (1.3), we have

$$\begin{aligned}
|L_n^{(\alpha)}(f; x)| &= \left| \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) \right| \leq \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \left| f\left(\frac{k}{n}\right) \right| \\
&\leq \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \sup |f(x)| = \|f\|.
\end{aligned}$$

\square

4. DIRECT RESULTS

Let $C_B(I)$ be the space of all the real valued continuous and bounded functions f on the interval I , endowed with the norm

$$\|f\| = \sup_{x \in I} |f(x)|.$$

For $f \in C_B(I)$, the Peetre's K -functional is defined by

$$K_2(f; \delta) := \inf_{g \in C_B^2(I)} \{\|f - g\| + \delta \|g''\|\}, \quad \delta > 0,$$

where $C_B^2(I) = \{g \in C_B(I) : g', g'' \in C_B(I)\}$. By DeVore and Lorentz ([5], p.177. Theorem 2.4) there exists an absolute constant $C > 0$ such that

$$K_2(f; \delta) \leq C \omega_2(f; \sqrt{\delta}), \quad (4.1)$$

where $\omega_2(f; \sqrt{\delta})$ is second order modulus of continuity defined by

$$\omega_2(f; \sqrt{\delta}) = \sup_{0 < h < \sqrt{\delta}} \sup_{x \in I} |f(x+2h) - 2f(x+h) + f(x)|.$$

Also the first order modulus of smoothness (or simply modulus of continuity) is given by

$$\omega(f; \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in I} |f(x+h) - f(x)|.$$

Theorem 4.1. *For $f \in C_B(I)$, we have*

$$\left| L_n^{(\alpha)}(f; x) - f(x) \right| \leq \omega\left(f, \frac{px + nx(\lambda+1)\alpha}{x(1-(\lambda+1)\alpha)}\right) + C \omega_2\left(f, \frac{\sqrt{\psi_{n,\lambda}^{(\alpha)}(x)}}{2}\right),$$

where C is a positive constant and

$$\psi_{n,\lambda}^{(\alpha)}(x) = L_n^{(\alpha)}((t-x)^2; x) + \left\{ \frac{px + nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)} \right\}^2.$$

Proof. First we consider auxiliary operators

$$\hat{L}_n^{(\alpha)}(f; x) = L_n^{(\alpha)}(f; x) + f(x) - f\left(\frac{n+p}{n} \cdot \frac{x}{1-(\lambda+1)\alpha}\right). \quad (4.2)$$

For all $x \in I$ we observed that $\hat{L}_n^{(\alpha)}(f; x)$ are linear such that

$$\hat{L}_n^{(\alpha)}(1; x) = 1 \quad \text{and} \quad \hat{L}_n^{(\alpha)}(t; x) = x,$$

i. e., preserve linear functions. Therefore

$$\hat{L}_n^{(\alpha)}(t-x; x) = 0. \quad (4.3)$$

Let $g \in C_B^2(I)$ and $t, x \in I$ then Taylor's theorem implies

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

we can write

$$\begin{aligned}
\hat{L}_n^{(\alpha)}(g; x) - g(x) &= g'(x) \hat{L}_n^{(\alpha)}((t-x); x) + \hat{L}_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du; x\right) \\
&= \hat{L}_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du; x\right) \\
&= L_n^{(\alpha)}\left(\int_x^t (t-u)g''(u)du; x\right) \\
&\quad - \int_x^{\frac{n+p}{n}\left(\frac{x}{1-(\lambda+1)\alpha}\right)} \left(\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha} - u\right) g''(u)du.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\left|\hat{L}_n^{(\alpha)}(g; x) - g(x)\right| &\leq L_n^{(\alpha)}\left(\left|\int_x^t (t-u)g''(u)du\right|; x\right) \\
&\quad + \left|\int_x^{\frac{n+p}{n}\left(\frac{x}{1-(\lambda+1)\alpha}\right)} \left(\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha} - u\right) g''(u)du\right|. \quad (4.4)
\end{aligned}$$

Since $\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2 \|g''\|$ and

$$\left|\int_x^{\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha}} \left(\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha} - u\right) g''(u)du\right| \leq \left\{\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha} - x\right\}^2 \|g''\|.$$

Therefore (4.4) implies that

$$\begin{aligned}
\left|\hat{L}_n^{(\alpha)}(g; x) - g(x)\right| &\leq \left[L_n^{(\alpha)}\left((t-x)^2; x\right) + \left\{\frac{n+p}{n}\left(\frac{x}{1-(\lambda+1)\alpha}\right) - x\right\}^2\right] \|g''\| \\
&\leq \left[L_n^{(\alpha)}\left((t-x)^2; x\right) + \left\{\frac{px + nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right\}^2\right] \|g''\| \\
&= \psi_{n,\lambda}^{(\alpha)}(x) \|g''\|, \quad (4.5)
\end{aligned}$$

Again using definition of auxiliary operators and from Lemma 3.5, we get

$$\begin{aligned}
\left|L_n^{(\alpha)}(f; x) - f(x)\right| &\leq \left|\hat{L}_n^{(\alpha)}((f-g); x)\right| + |g(x) - f(x)| + \left|\hat{L}_n^{(\alpha)}(g; x) - g(x)\right| \\
&\quad + \left|f\left(\frac{n+p}{n}\frac{x}{1-(\lambda+1)\alpha}\right) - f(x)\right| \\
&\leq 4\|f-g\| + \psi_{n,\lambda}^{(\alpha)}(x) \|g''\| + \omega\left(f; \frac{px + nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right).
\end{aligned}$$

Taking infimum on both the sides over $g \in C_B^2(I)$,

$$\left|L_n^{(\alpha)}(f; x) - f(x)\right| \leq 4K_2 \left(f; \frac{\psi_{n,\lambda}^{(\alpha)}(x)}{4}\right) + \omega\left(f; \frac{px + nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right).$$

Hence by (4.1), we get

$$\left|L_n^{(\alpha)}(f; x) - f(x)\right| \leq C\omega_2\left(f; \frac{\sqrt{\psi_{n,\lambda}^{(\alpha)}(x)}}{2}\right) + \omega\left(f; \frac{px + nx(\lambda+1)\alpha}{n(1-(\lambda+1)\alpha)}\right).$$

□

We consider the following Lipschitz-type space (see [20])

$$Lip_M^*(\beta) := \left\{ f \in C_B(I) : |f(y) - f(x)| \leq M \frac{|y - x|^\beta}{(x + y)^{\beta/2}}; x, y \in (0, \infty) \right\},$$

where M is a positive constant and $0 < \beta \leq 1$.

Theorem 4.2. For all $x \in I$ and $f \in Lip_M^*(\beta)$, $0 < \beta \in (0, 1]$ we get

$$\left| L_n^{(\alpha)}(f; x) - f(x) \right| \leq M \left(\frac{\phi_n^{(\alpha)}(x)}{x} \right)^{\beta/2}, \quad (4.6)$$

where $\phi_n^{(\alpha)}(x) = L_n^{(\alpha)}((t - x)^2; x)$.

Proof. Assume that $\beta = 1$. Then, for $f \in Lip_M^*(1)$, we have

$$\begin{aligned} \left| L_n^{(\alpha)}(f; x) - f(x) \right| &\leq \left| \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right) - f(x) \right| \\ &\leq \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \left| f\left(\frac{k}{n}\right) - f(x) \right| \leq M \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \frac{\left| \frac{k}{n} - x \right|}{\left(\frac{k}{n} + x \right)^{1/2}}. \end{aligned}$$

Applying Cauchy-Schwarz inequality for sum and $\frac{1}{\sqrt{\frac{k}{n} + x}} \leq \frac{1}{\sqrt{x}}$, we have

$$\begin{aligned} \left| L_n^{(\alpha)}(f; x) - f(x) \right| &\leq \frac{M}{\sqrt{x}} \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \left\{ \left(\frac{k}{n} - x \right)^2 \right\}^{1/2} \\ &\leq \frac{M}{\sqrt{x}} \left\{ \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \right\}^{1/2} \left\{ \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)} \left(\frac{k}{n} - x \right)^2 \right\}^{1/2} \\ &\leq \frac{M}{\sqrt{x}} \left\{ L_n^{(\alpha)}(1; x) \right\}^{1/2} \left\{ L_n^{(\alpha)}((t - x)^2; x) \right\}^{1/2} = M \left\{ \frac{\phi_n^{(\alpha)}(x)}{x} \right\}^{1/2}. \end{aligned}$$

Therefore result is true for $\beta = 1$.

Now we prove the required result for $0 < \beta < 1$. Consider $f \in Lip_M^*(\beta)$

$$\left| L_n^{(\alpha)}(f; x) - f(x) \right| \leq \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \left| f\left(\frac{k}{n}\right) - f(x) \right| \leq M \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \frac{\left| \frac{k}{n} - x \right|^\beta}{\left(\frac{k}{n} + x \right)^{\beta/2}}.$$

Using Holder's inequality for sum with $p = 2/\beta, q = 2/(2 - \beta)$ and inequality $\frac{1}{\sqrt{\frac{k}{n} + x}} \leq \frac{1}{\sqrt{x}}$, we have

$$\begin{aligned} \left| L_n^{(\alpha)}(f; x) - f(x) \right| &\leq \frac{M}{x^{\beta/2}} \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \left\{ \left(\frac{k}{n} - x \right)^2 \right\}^{\beta/2} \\ &\leq \frac{M}{x^{\beta/2}} \left\{ \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \left(\frac{k}{n} - x \right)^2 \right\}^{\beta/2} \left\{ \sum_{k=0}^{\infty} \omega_{n,k}^{(\alpha)}(x) \right\}^{\frac{2-\beta}{2}} \\ &\leq M \left\{ \frac{L_n^{(\alpha)}((t - x)^2; x)}{x} \right\}^{\beta/2} = M \left\{ \frac{\phi_n^{(\alpha)}(x)}{x} \right\}^{\beta/2}. \end{aligned}$$

□

5. WEIGHTED APPROXIMATION

Gadjiev [8, 9] studied the weight spaces $C_\varphi(I)$ and $B_\varphi(I)$ of real valued functions defined on I with $\varphi(x) = 1 + x^2$ and proved that Korovkin's theorem in general does not hold on these spaces. Here

$$B_\varphi(I) := \{f : |f(x)| \leq M_f \varphi(x)\},$$

with

$$\|f\|_\varphi = \sup_{x \in I} \frac{|f(x)|}{\varphi(x)},$$

and

$$C_\varphi(I) := \{f : f \in B_\varphi(I) \text{ and } f \text{ continuous}\},$$

i.e. $C_\varphi(I) = C(I) \cap B_\varphi(I)$ is the subspace of $B_\varphi(I)$ containing continuous functions and

$$C_\varphi^*(I) := \left\{ f \in C_\varphi(I) : \lim_{x \rightarrow \infty} \frac{|f(x)|}{\varphi(x)} < \infty \right\}.$$

But Korovkin's theorem holds in the space $C_\varphi^*(I)$.

The usual modulus of continuity of f on $[0, b]$ is defined as:

$$\omega_b(f, \delta) = \sup_{|t-x| \leq \delta} \sup_{x, t \in [0, b]} |f(t) - f(x)|.$$

Theorem 5.1. *Let $f \in C_\varphi(I)$ then for operators $L_n^{(\alpha)}(f; x)$ we have*

$$\left\| L_n^{(\alpha)}(f; x) - f(x) \right\|_{C[0, b]} \leq 4M_f(1 + b^2) \phi_n^{(\alpha)}(b) + 2\omega_{b+1}\left(f, \sqrt{\phi_n^{(\alpha)}(b)}\right), \quad (5.1)$$

where $\phi_n^{(\alpha)}(x) = L_n^{(\alpha)}((t-x)^2; x)$.

Proof. Let $x \in [0, b]$, $t \in (b+1, \infty)$, and $t-x > 1$ we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f \varphi(t-x) = M_f \{1 + (t-x)^2\} = M_f(1 + t^2 + x^2 - 2xt) \\ &\leq M_f(2 + t^2 + x^2) = M_f \{2 + 2x^2 + (t-x)^2 + 2x(t-x)\} \\ &\leq M_f(t-x)^2 \{3 + 2x + 2x^2\} \leq 4M_f(t-x)^2(1 + x^2) \\ &\leq 4M_f(t-x)^2(1 + b^2). \end{aligned} \quad (5.2)$$

If $x \in [0, b]$ and $t \in [0, b+1]$ then we have

$$|f(t) - f(x)| \leq \omega_{b+1}(|t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{b+1}(f, \delta), \delta > 0. \quad (5.3)$$

Combining (5.2) and (5.3), we obtain

$$|f(t) - f(x)| \leq 4M_f(1 + b^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right) \omega_{b+1}(f, \delta).$$

Using Cauchy Schwartz inequality we get

$$\begin{aligned}
\left| L_n^{(\alpha)}(t) - f(x) \right| &\leq 4M_f (1 + b^2) L_n^{(\alpha)} \left((t - x)^2; x \right) \\
&\quad + \left(1 + \frac{1}{\delta} L_n^{(\alpha)}(|t - x|; x) \right) \omega_{b+1}(f, \delta) \\
&\leq 4M_f (1 + b^2) L_n^{(\alpha)} \left((t - x)^2; x \right) \\
&\quad + \left[1 + \frac{1}{\delta} \left\{ L_n^{(\alpha)} \left((t - x)^2; x \right) \right\}^{1/2} \right] \omega_{b+1}(f, \delta) \\
&\leq 4M_f (1 + b^2) \phi_n^{(\alpha)}(b) + 2\omega_{b+1} \left(f, \sqrt{\phi_n^{(\alpha)}(b)} \right).
\end{aligned}$$

□

Theorem 5.2. *Let $f \in C_\varphi^*(I)$ then, we have*

$$\lim_{n \rightarrow \infty} \left\| L_n^{(\alpha)}(f; x) - f(x) \right\|_\varphi = 0. \quad (5.4)$$

Proof. From [9], it's sufficient to verify the following three equations

$$\lim_{n \rightarrow \infty} \left\| L_n^{(\alpha)}(t^i; x) - t^i \right\|_\varphi = 0, \quad i = 0, 1, 2. \quad (5.5)$$

Clearly equation (5.5) holds for $i = 0$ as $L_n^{(\alpha)}(1; x) = 1$. Now using Lemma 3.2 we have

$$\begin{aligned}
\left\| L_n^{(\alpha)}(t; x) - x \right\|_\varphi &= \sup_{x \in I} \frac{1}{\varphi(x)} \left| \frac{n+p}{n} \frac{x}{(1 - \bar{\lambda} + 1\alpha)} - x \right| \\
&= \sup_{x \in I} \frac{x}{\varphi(x)} \left| \frac{p + \bar{\lambda} + 1\alpha n}{n(1 - \bar{\lambda} + 1\alpha)} \right| \leq \frac{p + \bar{\lambda} + 1\alpha n}{n(1 - \bar{\lambda} + 1\alpha)},
\end{aligned}$$

which implies that $\lim_{n \rightarrow \infty} \left\| L_n^{(\alpha)}(t; x) - x \right\|_\varphi = 0$. Similarly we have

$$\begin{aligned}
\left\| L_n^{(\alpha)}(t^2; x) - x^2 \right\|_\varphi &= \sup_{x \in I} \frac{1}{\varphi(x)} \left| \frac{n+p}{n^2} \frac{x}{(1 - \lambda\alpha)(1 - \bar{\lambda} + 1\alpha)} \right. \\
&\quad \times \left[\frac{(n+p+\lambda+1)x(x+\alpha)}{1 - 2\bar{\lambda} + 1\alpha} + x(1 + \lambda x) \right] - x^2 \Big| \\
&\leq \left| \frac{n+p}{n^2} \frac{1}{(1 - \lambda\alpha)(1 - \bar{\lambda} + 1\alpha)} \times \left[\frac{(n+p+\lambda+1)(1+\alpha)}{1 - 2\bar{\lambda} + 1\alpha} + \lambda + 1 \right] - 1 \right|.
\end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \left\| L_n^{(\alpha)}(t^2; x) - x^2 \right\|_\varphi = 0$.

This completes the proof. □

Finally we give the rate of convergence for the generalized positive linear operators (1.3) with the help of Yüksel and Ispir [26] works. For every $f \in C_\varphi^*(I)$, the weighted modulus of continuity is defined by:

$$\Omega(f; \delta) = \sup_{x \in I, 0 < h \leq \delta} \frac{|f(x+h) - f(x)|}{1 + (x+h)^2}.$$

Properties of the weighted modulus of continuity are given by the following lemma:

Lemma 5.1. [26] Let $f \in C_\varphi^*(I)$ then we have

- (i) $\Omega(f; \delta)$ is a monotone increasing function of δ ;
- (ii) $\lim_{\delta \rightarrow 0+} \Omega(f; \delta) = 0$;
- (iii) for each $m \in \mathbb{N}$, $\Omega(f; m\delta) = m\Omega(f; \delta)$;
- (iv) for each $\mu \in I$, $\Omega(f; \mu\delta) = (1 + \mu)\Omega(f; \delta)$.

Theorem 5.3. Let $f \in C_\varphi^*(I)$ then for operators $L_n^{(\alpha)}(f; x)$ we have

$$\sup_{x \in I} \frac{|L_n^{(\alpha)}(f; x) - f(x)|}{(1+x^2)^{5/2}} \leq K\Omega\left(f; \frac{1}{\sqrt{n}}\right),$$

where K is positive constant.

Proof. For $x, t \in [0, \infty)$, $\delta > 0$ and by definition of $\Omega(f; \delta)$ and Lemma 3.4, we have

$$\begin{aligned} |f(t) - f(x)| &\leq \left(1 + (x + |t - x|)^2\right) \Omega(f; |t - x|) \\ &\leq \left(1 + (x + |t - x|)^2\right) \left(1 + \frac{|t - x|}{\delta}\right) \Omega(f; \delta) \\ &\leq 2(1 + x^2) \left(1 + (t - x)^2\right) \left(1 + \frac{|t - x|}{\delta}\right) \Omega(f; \delta). \end{aligned}$$

Using the fact that $L_n^{(\alpha)}(f; x)$ are positive linear operators, we get

$$\begin{aligned} |L_n^{(\alpha)}(f; x) - f(x)| &\leq \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \left|f\left(\frac{k}{n}\right) - f(x)\right| \\ &\leq 2\Omega(f; \delta) (1 + x^2) \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \left\{1 + \left(\frac{k}{n} - x\right)^2\right\} \left(1 + \frac{|\frac{k}{n} - x|}{\delta}\right) \\ &\leq 2\Omega(f; \delta) (1 + x^2) \left[\sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) + \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \left(\frac{k}{n} - x\right)^2 \right. \\ &\quad \left. + \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \frac{|\frac{k}{n} - x|}{\delta} + \sum_{k=0}^{\infty} w_{n,k}^{(\alpha)}(x) \left(\frac{k}{n} - x\right)^2 \frac{|\frac{k}{n} - x|}{\delta} \right] \\ &\leq 2\Omega(f; \delta) (1 + x^2) \left[1 + L_n^{(\alpha)}\left((t - x)^2; x\right) \right. \\ &\quad \left. + L_n^{(\alpha)}\left(\left(1 + (t - x)^2\right) \frac{|t - x|}{\delta}; x\right) \right]. \end{aligned} \quad (5.6)$$

Applying Cauchy Schwarz inequality and in view of Remark 3.1, we obtain

$$\begin{aligned} |L_n^{(\alpha)}(f; x) - f(x)| &\leq 2\Omega(f; \delta) (1 + x^2) \left[1 + L_n^{(\alpha)}\left((t - x)^2; x\right) \right. \\ &\quad \left. + \frac{1}{\delta} \left\{ \left(L_n^{(\alpha)}\left((t - x)^2; x\right)\right)^{1/2} + \left(L_n^{(\alpha)}\left((t - x)^2; x\right)\right)^{1/2} \left(L_n^{(\alpha)}\left((t - x)^4; x\right)\right)^{1/4} \right\} \right] \\ &\leq K(1 + x^2)^{5/2} \Omega\left(f; \frac{1}{\sqrt{n}}\right), \end{aligned}$$

where $\delta = \frac{1}{\sqrt{n}}$ and $K = 2(1 + A_1 + \sqrt{A_1} + \sqrt{A_1 A_2})$. Hence the result. \square

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